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Real and imaginary mass generation in the presence of external fields and axions

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Abstract: Small fluctuations around a constant electric or constant magnetic field F are analyzed in a theory with pseudo scalar ϕ with a coupling $g\phi F\tilde{F}$. It is found that a magnetic external field leads to mass generation for the small perturbations, while an electric field suffers from a tachyonic mass generation in the case in which the field strength is higher than a critical value (related to the pseudo scalar mass). The vacuum energy can be exactly evaluated and it is found that an imaginary part is present when the external electric field exceeds its critical value.

Keywords: Nonperturbative Effects, Spontaneous Symmetry Breaking.

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1. Introduction

1.1 Preliminary considerations

A dynamical mechanism providing mass to vector gauge bosons is instrumental to match theoretical models of fundamental interactions with the particle spectrum observed in high energy experiments. A blueprint of dynamical mass generation is given by the Schwinger Model, or QED_2, where fermions quantum fluctuations induce a mass term for the two-dimensional photon. Extension of this non-perturbative quantum effect to four-dimensional gauge theories has still to come because the gauge field effective action cannot be computed exactly in $4D$. In the meanwhile, the archetypal mechanism for gauge field theory mass generation is Spontaneous Symmetry Breaking, induced either by classical tachyonic mass terms [1] or by quantum radiative corrections [2]. The Coleman-Weinberg breaking of gauge symmetry avoids classical tachyonic mass terms and gives raise to a non-vanishing vacuum expectation value for massless scalar fields through radiative quantum corrections. In this paper we are going to discuss a "complementary" mechanism, where mass follows from the breaking of rotational invariance induced by a classical background configuration of the gauge field strength. A real, or tachyonic, mass is obtained according with the magnetic, or electric, nature of the background field. The model implementing this effect consists of a scalar field ϕ non-minimally coupled to a $U(1)$ gauge vector (non-abelian extensions of this model are planned for future investigations) through an interaction term of the form:

$$\mathcal{L}_{\rm I} = \frac{g}{8}\phi\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}\,. \tag{1.1}$$

The interaction term $\mathcal{L}_{\rm I}$ has a long history dating back to the celebrated ABJ anomaly and neutral pion electromagnetic decay [3]. Moreover by keeping fixed the form and varying the strength of the coupling constant, $\mathcal{L}_{\rm I}$ is equally well suited to describe the axion field currently appearing in many astrophysical and quantum field theoretical problems [4].

In what follows we are going to analyze the case in which the electromagnetic field is a purely electric or purely magnetic background, with special interest about the dynamics of its fluctuations. Before embarking this program and before giving a more detailed account of the main aspects of our approach, it is worth to recall some important steps already taken in the past in this direction. In the main part of this work we will stress more carefully analogies, as well as differences, with what we are proposing in this paper. In particular the fact that an external *magnetic* field modifies the dispersion relation of photons coupled to (pseudo)scalars was already discussed, for example, in [5]; there the authors have in mind an experimental set-up for the detection of pseudoscalars coupled to two photons based on the fact that the photon effective mass provided by the pseudoscalar coupling is responsible for an ellipticity in an initially linearly polarized beam.[1] Concerning the situation in which a background electric field is present, recently this problem has been analyzed in [8], where the authors show under which conditions an external electric field decays to pseudoscalars and discuss some particular configurations in which their results can be applied. Postponing a deeper analysis to what follows, we think that an important point to be stressed already at this early stage, is the fact that in the above studies the discussion is *perturbative* whereas, in the present paper, we are going to analyze a second order effective approximation for the dynamics of the fluctuations of the electromagnetic field, only after a full, *non-perturbative* treatment of the pseudoscalar.

Before developing this part, we will shortly present some interesting features of the model in a *naive* form. We remember that, indeed, it is a special feature of the coupling (1.1) to generate physical masses, or tachyonic instabilities, when an appropriate classical configuration of the scalar field ϕ is turned on: performing an integration by parts in the action associated with (1.1), we end up with an interaction term of the form

$$\mathcal{L}_{\rm I} = \frac{g}{8}\left(\partial_\beta\phi\right)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}A_\alpha\,. \tag{1.2}$$

We can then consider a background configuration selecting a preferred spacelike direction, e.g. $\langle\phi\rangle = \text{const.}\cdot\delta^3_\mu x^\mu$, so that (1.2) gives a $(2+1)$-dimensional, Chern–Simons type,

[1] This set-up is now a fully working project, PVLAS, at the I.N.F.N. Legnaro laboratories, taking data since May 1999: about this we refer the interested reader to [6] and references therein, as well as to [7] for a more direct connection with axion experiments.

mass term [9]. The resulting massive Chern–Simons model is embedded into a $(3+1)$-dimensional theory. Thus, rotational invariance in embedding space is broken [10]. It is interesting to compare this result with the case in which the background field is time dependent only, i.e. $\langle \phi \rangle \equiv \varphi(t)$. The resulting Chern–Simons model is endowed with a tachyonic mass term [11], meaning that this type of background field is unstable. This kind of instabilities could play a role in baryogenesis, as it has been argued in [11], [12]. Moreover, a time dependent axion field may also affect the growth of primordial magnetic fields [13], produce effective Lorentz and parity violating modifications of electrodynamics and affect the polarization of radiation coming from distant galaxies [14].

1.2 A "naive" demonstrative approach

In this paper we want to explore "the other side of the Moon", that is what happens if the gauge field strength $F_{\mu\nu}$, and not the scalar/axion field ϕ, acquires a non-vanishing background value. As an example, suppose $\langle F_{\mu\nu} \rangle = B\delta_{[\mu|2}\delta_{\nu]3}$, i.e. $\langle F_{\mu\nu} \rangle$ is a constant magnetic field along the x^3 direction. This choice is not *ad hoc* as it could appear: we are trying to mimic the QCD vacuum, where a constant, color, magnetic field lowers the energy density with respect to the perturbative Fock vacuum, where no gluons are present [15]. By introducing the proper kinetic term for ϕ and the fluctuation field $f_{\mu\nu}$, we see that (1.2) turns into

$$\mathcal{L} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{g}{4} B \phi \epsilon_{23\alpha\beta} f^{\alpha\beta} \,. \tag{1.3}$$

Equation (1.3) shows as ϕ couples only to the f^{01} component of the fluctuation field strength. Accordingly, we can write the following effective lagrangian in the $(0-1)$-plane:

$$\mathcal{L}_{01} \equiv -\frac{1}{2} f_{01} f^{01} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{g}{4} B \phi f^{01} \,. \tag{1.4}$$

By freezing the x^2, x^3 dependence of the fields in (1.4) we get the bosonized Schwinger model [16] and solving the classical field equation for ϕ we obtain the non-local effective action for the dimensionally reduced, massive, electromagnetic field

$$\mathcal{L}_{01} \equiv -\frac{1}{2} f_{01} f^{01} - \frac{g^2 B^2}{32} f^{01} \frac{1}{\Box} f_{01} \,, \tag{1.5}$$

where the generated mass square is proportional to $g^2 B^2$. Of course the full spectrum of excitations will contain more general types of spacetime dependence, so that we expect here an anisotropic mass generation effect.

We can repeat the same calculations with an electric background, e.g.

$$\langle F_{\mu\nu} \rangle = E \delta_{[\mu|0} \delta_{1|\nu]} \,.$$

Then, instead of (1.5), we obtain for the small excitations in the $(2-3)$-plane

$$\mathcal{L} = -\frac{1}{2} f_{23} f^{23} + \frac{g^2 E^2}{32} f^{23} \frac{1}{\Box} f_{23} \,. \tag{1.6}$$

By comparing (1.6) with (1.5), we see that in the electric case there is a different sign in front of the non-local term, i.e. the generated mass is tachyonic. This situation for the electric background can be partially cured providing a non vanishing mass $m_{\rm A}$ to the axion field, so that the tachyonic mass generation sets in only around the critical electric field $gE \sim m_{\rm A}$.

As we will see in the next sections, the theory can be solved beyond the two dimensional truncated sectors mentioned here. Furthermore, it is possible to compute the vacuum energy resulting from small fluctuations around the chosen background. The vacuum energy develops an imaginary part for electric fields bigger than some critical value, of the order of $m_{\rm A}/g$. It can be worth now to remark a difference with respect to the instability in two dimensional QED. Both effects are triggered by an external electric field, but the standard Schwinger mechanism operates for any value of the constant electric field, provided it extends to sufficiently large distances. Even with a weak field, one can perform enough work to create a new pair. By contrast, in our case, the effect is present only for field strengths bigger than a critical one, regardless of whether it extends over a large region of space or not.

The paper is organized as follows. In subsection 2.1 we set up the "sum over histories" formulation for the system under study: in particular we show how an effective lagrangian for small fluctuations around a pure electric/magnetic background can be obtained, as a quadratic approximation, after integrating out exactly (and *non-perturbatively*) the scalar field and neglecting higher order terms. The next subsection 2.2 deals in more detail with the mass generation effect, fully exploiting the difference between the electric and the magnetic case and determining the propagator and its spectrum in momentum space. Then in section 3 the expression of the vacuum energy is introduced together with the exact expression for the propagator. The main steps in the computation of the free energy are outlined in section 4 and the emergence of an imaginary part in the pure electric case is analyzed. Discussion and conclusions are in section 5, with particular emphasis about the relation between the tachyonic mass-generation effect and a possible instability of homogeneous electric fields above some threshold. Three appendices follow, where technical details can be found about the solution of the eigenvalue equation for the momentum space propagator (appendix A), the determination of the propagator itself (appendix B) and the explicit computation of the free energy (appendix C), with particular attention at the behavior in the infrared and ultraviolet limits.

2. General analysis of the small perturbations

2.1 Path integral

An effective technical framework to investigate quantum fluctuations around an external background configuration is provided by the Feynman path integral formalism. The partition function(al) encoding the dynamics of the interacting ϕ and A_μ fields reads

$$Z \equiv \int [\mathcal{D}\phi][\mathcal{D}A] \exp\left\{-\imath \int d^4x \mathcal{L}\right\}, \tag{2.1}$$

where

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}\, F^{\mu\nu} + \frac{g}{8}\, \phi\, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}\, F_{\rho\sigma} + \frac{1}{2}\partial_\mu\phi\, \partial^\mu\phi + \frac{m_{\rm A}^2}{2}\phi^2 \tag{2.2}$$

and both the gauge fixing and ghost terms are, momentarily, understood in the functional measure $[\mathcal{D}A]$. For the sake of generality, we assigned a non-vanishing mass to the pseudo scalar field ϕ. Since the path integral (2.1) is gaussian in ϕ, the scalar field can be integrated away exactly, i.e.

$$\int[\mathcal{D}\phi]\exp\left\{-\imath\int d^4x\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+\frac{g}{8}\phi\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}+\frac{m_{\rm A}^2}{2}\phi^2\right]\right\}=$$
$$=\left[\det\left(\frac{\Box+m_{\rm A}^2}{\mu^2}\right)\right]^{-1/2}\exp\left\{\imath\frac{g^2}{128}\int d^4x\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\frac{1}{\Box+m_{\rm A}^2}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}\right\}, \quad (2.3)$$

where μ is a mass scale coming from the definition of the measure $[\mathcal{D}\phi]$. Integrating out the ϕ field induces a non-local effective action for the A field. We stress that this effective action, being obtained at the *non-perturbative* level, takes into account (as an effective action for A) all the effects due to the presence of the pseudoscalars at all perturbative orders. This is a crucial difference with many of the previous works on the subject. As is clear from (2.3), the resulting path integral is quartic in A but, even if it cannot be computed in a closed form, the background field method provides a reliable approximation scheme to deal with this problem. We thus split $F_{\mu\nu}$ in the sum of a classical background $\langle F_{\mu\nu}\rangle$ and a small fluctuation $f_{\mu\nu}$:

$$F_{\mu\nu}=\langle F_{\mu\nu}\rangle+f_{\mu\nu}\,. \quad (2.4)$$

In the case of a pure electric or a pure magnetic background we have

$$\epsilon^{\mu\nu\alpha\beta}\langle F_{\mu\nu}\rangle\langle F_{\alpha\beta}\rangle=0\,. \quad (2.5)$$

Thus, expanding $\mathcal{L}$ up to quadratic terms and dropping a total divergence, we obtain for the effective lagrangian of the fluctuations of the electromagnetic field

$$\mathcal{L}^{(2)}=-\frac{1}{4}\langle F_{\mu\nu}\rangle\langle F^{\mu\nu}\rangle-\frac{1}{4}f_{\mu\nu}f^{\mu\nu}-\frac{g^2}{16}\epsilon^{\mu\nu\alpha\beta}\langle F_{\mu\nu}\rangle\epsilon^{\rho\sigma\gamma\delta}\langle F_{\rho\sigma}\rangle f_{\alpha\beta}\frac{1}{\Box+m_{\rm A}^2}f_{\gamma\delta}\,. \quad (2.6)$$

As we will see in a while, there is an important sign difference between the magnetic and the electric case.

2.2 The signature of the mass generation: ordinary *versus* tachyonic

In this subsection we turn to the analysis of the contribution

$$\epsilon^{\mu\nu\alpha\beta}\langle F_{\mu\nu}\rangle\epsilon^{\rho\sigma\gamma\delta}\langle F_{\rho\sigma}\rangle \quad (2.7)$$

of equation (2.6), which will be responsible for ordinary or tachyonic mass generation according to whether $\langle F_{\mu\nu}\rangle$ represents an external magnetic or electric field. We notice that our results are not inconsistent with those that can be deduced from [8, equation (31)], from which it is clear that the sign of the contribution from the interaction lagrangian changes in the case of purely electric or purely magnetic background.

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Let us now start from the case of a constant magnetic field B. Then, without loosing generality, we can rotate the reference frame to align an axis, say x^1, with B. Accordingly, $\langle F_{\mu\nu}\rangle = B\delta_{[\mu|2}\delta_{\nu]3}$, so that

$$\epsilon^{\mu\nu\alpha\beta}\langle F_{\mu\nu}\rangle\epsilon^{\rho\sigma\gamma\delta}\langle F_{\rho\sigma}\rangle = 4B^2\epsilon^{23\alpha\beta}\epsilon^{23\gamma\delta}\,. \tag{2.8}$$

It is clear thus that α, β, γ and δ in (2.8) can take only the values 0, 1. Denoting by ${}^{(2)}\eta^{\alpha\beta}$ the 2×2 Minkowski tensor, we must then have that

$$\begin{aligned}\epsilon^{\alpha\beta 23}\epsilon^{\gamma\delta 23} &= A\left({}^{(2)}\eta^{\alpha\gamma(2)}\eta^{\beta\delta} - {}^{(2)}\eta^{\alpha\delta(2)}\eta^{\beta\gamma}\right)\\ &= A\left(P^{\alpha\gamma}_{(10)}P^{\beta\delta}_{(10)} - P^{\alpha\delta}_{(10)}P^{\beta\gamma}_{(10)}\right),\end{aligned} \tag{2.9}$$

where $P^{\alpha\gamma}_{(10)}$ is the projector onto the $(0-1)$-plane, i.e. $P^{\alpha\gamma}_{(10)} = {}^{(2)}\eta^{\alpha\gamma}$. Contracting in relation (2.9) the couples of indices $({}^{\alpha\gamma})$ and $({}^{\beta\delta})$ we obtain

$$\epsilon^{\alpha\beta 23}\epsilon_{\alpha\beta}{}^{23} = A(2\times 2-2) = 2A\,;$$

since in lowering the indices $\alpha\beta$ there is the time involved, $\epsilon^{\alpha\beta 23}\epsilon_{\alpha\beta}{}^{23} = -2$, so that $A=-1$.

Analogously in the electric case we can take $F_{\mu\nu} = E\delta_{[\mu|0}\delta_{1|\nu]}$. Then

$$\epsilon^{\alpha\beta\mu\nu}\langle F_{\mu\nu}\rangle\epsilon^{\gamma\delta\rho\sigma}\langle F_{\rho\sigma}\rangle = 4E^2\epsilon^{\alpha\beta 01}\epsilon^{\gamma\delta 01}\,, \tag{2.10}$$

with the indices α, β, γ, δ taking only the *spatial* values 2, 3. Now, following the same procedure and observing that no minus signs are involved in lowering spatial indices, we find

$$\begin{aligned}\epsilon^{\alpha\beta 01}\epsilon^{\gamma\delta 01} &= \left({}^{(2)}\delta^{\alpha\gamma(2)}\delta^{\beta\delta} - {}^{(2)}\delta^{\alpha\delta(2)}\delta^{\beta\gamma}\right)\\ &= P^{\alpha\gamma}_{(23)}P^{\beta\delta}_{(23)} - P^{\alpha\delta}_{(23)}P^{\beta\gamma}_{(23)}\,;\end{aligned}$$

again we introduced the projector notation $P^{\mu\nu}_{(23)} \equiv {}^{(2)}\delta^{\mu\nu}$, where ${}^{(2)}\delta^{\mu\nu}$ is the 2×2 Kronecker delta.

We can now write the equation of motion for the fluctuations, which from the action (2.6) turns out to be

$$\partial_\mu f^{\mu\beta} = 8g^2\epsilon^{\mu\nu\alpha\beta}\langle F_{\mu\nu}\rangle\frac{1}{\Box + m_{\mathrm{A}}^2}\epsilon^{\rho\sigma\gamma\delta}\langle F_{\rho\sigma}\rangle\partial_\alpha f_{\gamma\delta}\,. \tag{2.11}$$

The above equation can then be expressed, for both the electric and magnetic case, in Fourier space and in terms of the vector potential of the fluctuations, which we will call a_μ. Since $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, we have

$$k_\mu\left(k^\mu a^\nu - k^\nu a^\mu\right) = \frac{\kappa}{k^2 - m_{\mathrm{A}}^2}\left(\bar{k}^2\bar{g}^{\mu\nu} - \bar{k}^\mu\bar{k}^\nu\right)a_\mu\,. \tag{2.12}$$

Here and in what follows we define

$$\bar{g}^{\mu\nu} = P^{\mu\nu}_{(\cdot\cdot)} \tag{2.13}$$

$$\bar{k}^\mu = P^{\mu\alpha}_{(\cdot\cdot)}k_\alpha = \bar{g}^{\mu\alpha}k_\alpha \tag{2.14}$$

Eigenvalues	Eigenvectors
0	k_μ
$k^2 - \dfrac{\kappa \bar{k}^2}{k^2 - m_{\rm A}^2}$	$\tilde{k}_\mu = \begin{cases} \epsilon^{\mu\alpha 01} k_\alpha & \text{in the electric case} \\ \epsilon^{23\mu\alpha} k_\alpha & \text{in the magnetic case} \end{cases}$
k^2	$\tilde{k}^\perp_\mu = \epsilon_{\mu\nu\alpha\beta} \bar{k}^\nu \tilde{k}^\alpha k^\beta$
k^2	$E^\mu = \epsilon^{\mu\nu\rho\sigma} k_\nu \tilde{k}_\rho \tilde{k}^\perp_\sigma$

Table 1: Eigenvalues and eigenvectors of $\mathcal{D}^{-1^{\mu\nu}}(k)$.

and

$$P^{\mu\alpha}_{(\cdot\cdot)} = P^{\mu\alpha}_{(10)}\,, \qquad \kappa = +32 g^2 B^2$$

in the magnetic case, or

$$P^{\mu\alpha}_{(\cdot\cdot)} = P^{\mu\alpha}_{(23)}\,, \qquad \kappa = -32 g^2 E^2$$

in the electric case. Equation (2.12) can be written also as

$$\mathcal{D}^{-1^{\mu\nu}}(k) a_\mu = 0\,,$$

where we have defined

$$\mathcal{D}^{-1^{\mu\nu}}(k) = \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) - \frac{\kappa}{k^2 - m_{\rm A}^2}\left(\bar{k}^2 \bar{g}^{\mu\nu} - \bar{k}^\mu \bar{k}^\nu\right). \tag{2.15}$$

The operator in (2.15) can be diagonalized[2] as shown in appendix A. Four linearly independent physical states satisfy the eigenvector equation of $\mathcal{D}^{-1^{\mu\nu}}(k)$; we will call them k_μ, $\tilde{k}_\mu$, $\tilde{k}^\perp_\mu$, E^μ. Their definitions and corresponding eigenvalues are enlisted in table 1. There are, thus, two non-zero eigenvalues and one of them, k^2, has degeneracy two; moreover the "gauge" eigenvector is associated to the zero eigenvalue. This last results changes if we study the problem in the *covariant α-gauge*, when the eigenvector k^μ is then associated to a non-zero (but still background independent) eigenvalue, $1/\alpha$ (the corresponding inverse propagator will be called $\mathcal{D}^{-1^{\mu\nu}}(k;\alpha)$ as defined in what follows).

As promised above, we can now compare the obtained results with those derived in previous works on the subject. Again, we observe that a key point in our derivation is the (non-perturbative) path-integral procedure used to obtain equation (2.3): in this way all the effects due to the quantum fluctuations of the pseudoscalar field, at all orders, are taken into account. In particular, if we concentrate on the purely magnetic case, it is then clear that this constitutes a generalization of the results obtained in [5], where the secular equation is obtained considering plane wave solutions to the *classical* equations of

[2]We will summarize here the final result about eigenvectors and the corresponding eigenvalues with all the relevant definitions, referring the reader to the mentioned appendix for the detailed computation.

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motion associated to the Maxwell + Klein-Gordon action for the coupled electromagnetic and pseudoscalar fields. An analogous result for the electric case in presence of massless pseudoscalars can, for instance, be found in [17]: here the dispersion relations are obtained under physically very sensible restrictions but are, anyway, of perturbative character. Also the more detailed analysis of [18] uses a different kind of approximation with respect to the one employed in our calculation since the starting lagrangian in [18, equation (1)] is different from (2.6), our quadratic approximation to the full, non-perturbative, effective result in equation (2.3). We can thus trace back the differences between our results for the eigenvalues of the propagator and the one already derived in the literature on the subject, to the fact that we have taken into account the effects due to the pseduscalar fields in a substantially non-perturbative way.

3. Path integral quantization of small perturbations: propagator and vacuum energy

To compute the vacuum energy we proceed further in the covariantly quantized fashion we started in the previous section. In a covariant α-gauge the path integral for the partition function can be rewritten, using (2.15) and going to euclidean space, as

$$Z = \left[\det\left(\Box + m_{\mathrm{A}}^2\right)\right]^{-1/2} \int [\mathcal{D}f] \int [\mathcal{D}A_\mu]\delta\left[\partial_\mu A^\mu - f\right] e^{-\frac{1}{2\alpha}\int f^2 d^4x} \times \det\left[\partial^2\right] \times \\ \times \exp\left\{-\int \frac{d^4k}{(2\pi)^4} A_\mu(k) \left[\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) - \frac{\kappa\left(\bar{k}^2\bar{g}^{\mu\nu} - \bar{k}^\mu\bar{k}^\nu\right)}{k^2 - m_{\mathrm{A}}^2}\right] A_\nu(k)\right\}. \quad (3.1)$$

Here we have already performed the functional integration over ϕ, we remember that we are approximating the higher order lagrangian up to the second order in the fluctuations and understand the path-integrals in the euclidean sector. The functional integration over f can now be done and we get

$$Z = \frac{\det\left[\partial^2\right] \int [\mathcal{D}A_\mu] \exp\left\{-\int \frac{d^4k}{(2\pi)^4} A_\mu(k) \mathcal{D}^{-1\,\mu\nu}(k;\alpha) A_\nu(k)\right\}}{\left[\det\left(\Box + m_{\mathrm{A}}^2\right)\right]^{1/2}}, \quad (3.2)$$

where

$$\mathcal{D}^{-1\,\mu\nu}(k;\alpha) = \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) - \frac{\kappa}{k^2 - m_{\mathrm{A}}^2}\left(\bar{k}^2\bar{g}^{\mu\nu} - \bar{k}^\mu\bar{k}^\nu\right) + \frac{1}{\alpha}k^\mu k^\nu = \mathcal{D}^{-1\,\mu\nu}(k) + \frac{1}{\alpha}k^\mu k^\nu .$$

In order to solve (3.2) we have to find the eigenvalues, λ_k, of $\mathcal{D}^{-1\,\mu\nu}(k;\alpha)$: Z equals then the product of these eigenvalues. The requested eigenvalues are those related to the physical polarizations found in section 2.2, i.e. $\tilde{k}_\mu$, $\tilde{k}_\mu^\perp$, E_μ, with eigenvalues $k^2 - \kappa\bar{k}^2/(k^2 - m_{\mathrm{A}}^2)$, k^2 and k^2 respectively. As already discussed at the end of the previous section, the eigenvector k^μ has now a non-zero eigenvalue $1/\alpha$.

The vacuum energy, or free energy, is then

$$W = \ln Z = \frac{1}{2}\sum_j \ln\lambda_j = \frac{VT}{2}\sum\int \frac{d^4k}{(2\pi)^4}\ln(\lambda_k),$$

so that

$$W = \frac{VT}{2} \int \frac{d^4k}{(2\pi)^4} \ln \left(k^2 - \frac{\kappa \bar{k}^2}{k^2 - m_{\mathrm{A}}^2} \right) + VT \int \frac{d^4k}{(2\pi)^4} \ln \left(k^2 \right) + \\ + \frac{VT}{2} \int \frac{d^4k}{(2\pi)^4} \ln \left(k^2 - m_{\mathrm{A}}^2 \right) . \tag{3.3}$$

The propagator, i.e. the inverse of $\mathcal{D}^{-1\mu\nu}(k;\alpha)$, can also be found exactly, as shown in appendix B, and results to be

$$\mathcal{D}^{\mu\nu}(k;\alpha) = \frac{1}{k^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) + \frac{\kappa \bar{k}^2 \bar{g}^{\mu\nu}}{k^2 \left(k^2 \left(k^2 - m_{\mathrm{A}}^2 \right) - \kappa \bar{k}^2 \right)} + \\ - \frac{\kappa \bar{k}^\mu \bar{k}^\nu}{k^2 \left(k^2 \left(k^2 - m_{\mathrm{A}}^2 \right) - \kappa \bar{k}^2 \right)} + \alpha \frac{k^\mu k^\nu}{(k^2)^2} . \tag{3.4}$$

4. Evaluation of the vacuum energy and of its imaginary part

To compute the vacuum energy, apart from more standard contributions, we have to compute the following integral

$$I_0(k, \bar{k}; \kappa, m_{\mathrm{A}}) \equiv \int \frac{d^4k}{(2\pi)^4} \ln \left(k^2 - \frac{\kappa \bar{k}^2}{k^2 - m_{\mathrm{A}}^2} \right) . \tag{4.1}$$

It can be evaluated in closed form and we first rewrite it as

$$I_0(k, \bar{k}; \kappa, m_{\mathrm{A}}) = I(k, \bar{k}; \kappa, m_{\mathrm{A}}) - \int \frac{d^4k}{(2\pi)^4} \ln \left(k^2 - m_{\mathrm{A}}^2 \right) , \tag{4.2}$$

so that we can separate the second common contribution, from the first one, i.e.

$$I(k, \bar{k}; \kappa, m_{\mathrm{A}}) = \int \frac{d^4k}{(2\pi)^4} \ln \left[k^2 (k^2 - m_{\mathrm{A}}^2) - \kappa \bar{k}^2 \right] . \tag{4.3}$$

Using (4.2) for the right hand side of (4.1) appearing in (3.3), we get for the free energy

$$W = \frac{VT}{2} I(k, \bar{k}; \kappa, m_{\mathrm{A}}) + VT \int \frac{d^4k}{(2\pi)^4} \ln \left(k^2 \right) . \tag{4.4}$$

The computation of $I(k, \bar{k}; \kappa, m_{\mathrm{A}})$ is performed in appendix C. The infrared behavior is then extracted in appendix C.1. Ultraviolet divergences are regularized by a cut-off Λ and the leading contributions to the free energy are computed in appendix C.2. Thus the final result for the vacuum energy density is obtained multiplying (C.25) by $1/(4\pi^2)$ (because of (C.1)) and substituting for $I(k, \bar{k}; \kappa, m_{\mathrm{A}})$ in (4.4), where the second contribution can also be exactly evaluated. The final result is

$$W = \frac{VT}{8\pi^2} \left[I_\Lambda^{(4)} \Lambda^4 + I_{\ln\Lambda}^{(4)} \Lambda^4 \ln \Lambda + I_\Lambda^{(2)} \Lambda^2 + I_{\ln\Lambda}^{(2)} \Lambda^2 + I^{(0)} + I_{\ln\Lambda}^{(0)} \right] , \tag{4.5}$$

where the various contributions are defined at the end of appendix C. We will be especially interested in $VTI^{(0)}/(8\pi^2)$, which, according to equation (C.26), is

$$I^{(0)} = \frac{VT}{8\pi^2}\left[\frac{49\kappa^2+132\kappa m_{\mathrm{A}}^2+132m_{\mathrm{A}}^4}{576}+\frac{\kappa^2+3\kappa m_{\mathrm{A}}^2+3m_{\mathrm{A}}^4}{48}\ln 2-\right.$$
$$\left.-\frac{(\kappa+m_{\mathrm{A}}^2)^3}{24\kappa}\ln\left(\kappa+m_{\mathrm{A}}^2\right)+\frac{(m_{\mathrm{A}}^2)^3}{24\kappa}\ln\left(m_{\mathrm{A}}^2\right)\right]. \tag{4.6}$$

From this expression we see that the vacuum energy acquires an imaginary part in the case $\kappa+m_{\mathrm{A}}^2<0$, i.e. when the tachyonic modes are present. If we make use of the prescription $\kappa+m_{\mathrm{A}}^2\to\kappa+m_{\mathrm{A}}^2+\imath\epsilon$, the value of the imaginary part is

$$-\pi\frac{VT}{8\pi^2}\frac{\left(\kappa+m_{\mathrm{A}}^2\right)^3}{24\kappa}. \tag{4.7}$$

Notice that, as opposed to the real part, the imaginary part of the vacuum energy is cut-off independent. This is so because only the infrared region of the integrand in (4.3) contributes to the imaginary part.

We also observe that all contributions in (4.5) are finite in the limit of vanishing external field, i.e. when $\kappa\to 0$. This is immediately evident for the terms $I_{\Lambda}^{(4)}$, $I_{\ln\Lambda}^{(4)}$, $I_{\Lambda}^{(2)}$, $I_{\ln\Lambda}^{(2)}$ and $I_{\ln\Lambda}^{(0)}$, from their expressions at the end of appendix C. Concerning the contribution $I^{(0)}$, the two divergent terms of opposite sign in the second line of (4.6) give a finite contribution in the limit, as shown in (C.12).

The cut-off dependent parts are regularization dependent, as it is known from general experience with the regularization of divergent integrals. In particular subleading divergences (i.e. the Λ^2 terms in (4.5)) are highly dependent upon the regularization scheme and may vanish in certain of them. In the present problem, due to the complexity of the integrand in (4.3), other regularization schemes, like ζ-function regularization or dimensional regularization, are difficult to implement.

5. Discussion and conclusions

In this paper we have discussed effects of mass generation in an external magnetic field and of tachyonic mass generation in an external electric field in the case in which there is a pseudo scalar field with a pseudo scalar coupling $g\phi F_{\mu\nu}\tilde{F}^{\mu\nu}$. The effects due to the presence of the psedo scalars are *fully* considered: indeed in the derivation of an effective theory for the fluctuations of the electromagnetic field, the contributions from quantum fluctuations of the pseudo scalars are taken into account using a *non-perturbative* approach.

In the purely magnetic case the mass production for the electromagnetic fluctuations can be interpreted as in the cited works already present in the literature: the differences that we find in the eigenvalues of the propagator can be traced back to the non-perturbative character of our approach, as opposed to the perturbative analysis performed elsewhere.

A more careful discussion is instead required for the purely electric background, in connection with what we have called *tachyonic mass generation*. Indeed the appearance of an imaginary part in the free energy suggests the presence of an instability for homogeneous electric fields beyond some threshold due to pseudo scalar coupling. This means that the vacuum state must be redefined, to obtain a correct ground state for the theory. The analysis of what becomes the true ground state, a genuinely non perturbative effect, is beyond the scope of this manuscript. Nevertheless we would like, at least, to suggest a possible and simple, although incomplete, answer to this question. Indeed an analysis of Cornwall [22] indicates that in the case of a 3-dimensional, euclidean, tachyonic mass term (which in our discussion is generated by the time dependent scalar field of equation (1.2) but can have also another origin) the final result is the formation of an inhomogeneous state. Then, properly generalizing to our set-up the results described in [8], where inhomogeneous electromagnetic fields are shown to decay in pseudoscalars, it is not unreasonable to understand under what we have called *tachyonic mass generation* a real "tachyonic instability of the vacuum".

From the result of equation (4.6) it is possible to see that the free energy is well-defined in the limit of vanishing background fields ($\kappa \to 0$). Thus our effect is a genuinely non-perturbative one and does not relate to a particular choice of the regularization scheme. Moreover it is worth pointing out again that this "tachyonic instability of the vacuum" for fluctuations around a constant external electric field, is characterized by a threshold effect, i.e. the tachyonic mass generation is switched on for electric fields high enough, so that $\kappa + m_{\mathrm{A}}^2 < 0$. In the case of the neutral pion, we can obtain the value of the effective pion-photon coupling, defined by equation (1.2), from the observed value of the neutral pion lifetime [20] and from the value of the decay rate given the coupling (1.2) [21]. This gives us the values

$$\begin{aligned} g &= 2.53 \cdot 10^{-5}\,\mathrm{MeV}^{-1}\,, \\ m_\pi &= 134.97\,\mathrm{MeV}\,, \\ \Longrightarrow E_{\mathrm{crit.}} &= (1\,\mathrm{GeV})^2\,. \end{aligned} \tag{5.1}$$

This is a very high electric field, not available in normal laboratory conditions. Furthermore, if it were available, it would reveal the composite structure of the pion and the effective $g\phi F_{\mu\nu}\tilde{F}^{\mu\nu}$ coupling, used here, would not be applicable any more.

A different question would be then the study of this effect in the case of hypothetical axion particles. In this case, the threshold for the tachyonic mass generation to be set up becomes lower as lower values for the mass of the axion are considered.

Apart from the purely electric case, which is more subtle and, maybe, more *exciting* because of the *exotic* tachyonic mass term, it could be interesting a more detailed analysis of the magnetic case in connection with the set-up of PVLAS a presently running experiment at the Legnaro I.N.F.N. laboratories, near Venice, Italy.

Finally a totally different role for these effects, could be in the context of QCD. There, it is known that an external chromo-magnetic fields presents tachyonic instability. If we were to add a particle with coupling to $\epsilon^{\mu\nu\alpha\beta}F^a_{\mu\nu}F^a_{\alpha\beta}$ (this particle could represent a pseudo scalar bound state of quark and anti-quark pairs), we know that the effect of the external chromo-magnetic field together with the pseudo scalar coupling is of generating mass. The interplay of these two effects could then be an interesting subject for further research.

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A. Eigenvectors and eigenvalues of ${\mathcal{D}^{-1}}^{\mu\nu}(k)$

We will now find the solutions to the eigenvector equation

$$ {\mathcal{D}^{-1}}^{\mu\nu}(k) a_\nu = \lambda_k a^\mu \,, \tag{A.1} $$

where ${\mathcal{D}^{-1}}^{\mu\nu}(k)$ is defined in equation (2.15).

Firstly, there is a "trivial" gauge solution $a_\mu = H(k) k_\mu$, since k_μ is orthogonal to both terms enclosed in round brackets in the definition (2.15) of $\mathcal{D}^{-1}$; this can be seen from the relations

$$ \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) k_\mu = 0 $$

and

$$ \left(\bar{k}^2 \bar{g}^{\mu\nu} - \bar{k}^\mu \bar{k}^\nu\right) k_\mu = 0 $$

using the properties $\bar{g}^{\mu\nu} k_\nu = \bar{g}^{\mu\nu} \bar{k}_\nu$ and $\bar{k}^\nu k_\nu = \bar{k}^\nu \bar{k}_\nu$.

A second, non trivial polarization is $\tilde{k}^\mu = \epsilon^{\mu\alpha 01} k_\alpha$ in the electric case ($\tilde{k}^\mu = \epsilon^{23\mu\alpha} k_\alpha$ in the magnetic case), since $\tilde{k}^\mu k_\mu = 0 = \tilde{k}^\mu \bar{k}_\mu$ thanks to equations (2.13) and (2.14); $a_\mu = \tilde{k}_\mu$ is associated to the eigenvalue

$$ k^2 - \frac{\kappa \bar{k}^2}{k^2 - m_{\mathrm{A}}^2} \,. \tag{A.2} $$

Then, remaining independent physically relevant polarizations must be orthogonal to both k_μ and $\tilde{k}_\mu$, hence they can be parametrized as

$$ a^\mu = \epsilon^{\mu\nu\alpha\beta} d_\nu \tilde{k}_\alpha k_\beta \,; \tag{A.3} $$

since a^μ given by (A.3) is not affected by the "gauge transformation"

$$ d_\nu \longrightarrow d_\nu + \lambda_1 \tilde{k}_\nu + \lambda_2 k_\nu \,, $$

it then follows that only two components of d_ν are physically relevant. One of these, which we will call $\tilde{k}^\perp_\mu$, is obtained when $d_\nu = \bar{k}_\nu$: it is thus orthogonal to k^μ, $\bar{k}^\mu$, $\tilde{k}_\mu$ and corresponds to the eigenvalue k^2. This is the same eigenvalue of the last eigenvector, E^μ, which is given by

$$ E^\mu = \epsilon^{\mu\nu\rho\sigma} k_\nu \tilde{k}_\rho \tilde{k}^\perp_\sigma \,. $$

This can be verified by inserting it into (2.12) and observing that

$$\bar{k}^2\bar{g}_{\mu\nu} - \bar{k}_\mu\bar{k}_\nu \propto \tilde{k}_\mu\tilde{k}_\nu\,.$$

The above equality holds because

$$\bar{k}^2\bar{g}_{\mu\nu} - \bar{k}_\mu\bar{k}_\nu$$

is a projector onto the space orthogonal to $\bar{k}_\mu$ *in the* 2*-dimensional subspace*, where the direction orthogonal to $\bar{k}_\mu$ is nothing but $\tilde{k}_\nu$.

B. Determination of the propagator

To determine the propagator in the covariant α-gauge, $\mathcal{D}^{-1\mu\nu}(k;\alpha)$, with the four dimensional quantities that are at our disposal we consider the ansatz:

$$A\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + B\bar{g}^{\mu\nu} + C\bar{k}^\mu\bar{k}^\nu + D\frac{k^\mu k^\nu}{(k^2)^2}\,.$$

To determine the coefficients A, B, C, D we can now compute $\mathcal{D}^{-1\mu\nu}(k;\alpha)\mathcal{D}^{-1}{}_{\nu\alpha}(k;\alpha)$:

$$\begin{aligned}
\mathcal{D}^{-1\,\mu\nu}\mathcal{D}_{\nu\alpha} &= \left[\left(k^2g^{\mu\nu} - k^\mu k^\nu\right) - \frac{\kappa}{k^2 - m_{\mathrm{A}}^2}\left(\bar{k}^2\bar{g}^{\mu\nu} - \bar{k}^\mu\bar{k}^\nu\right) + \frac{1}{\alpha}k^\mu k^\nu\right] \times \\
&\quad \times \left[A\left(g_{\nu\alpha} - \frac{k_\nu k_\alpha}{k^2}\right) + B\bar{g}_{\nu\alpha} + C\bar{k}_\nu\bar{k}_\alpha + D\frac{k_\nu k_\alpha}{(k^2)^2}\right] \\
&= Ak^2\delta^\mu_\alpha - Ak^\mu k_\alpha - \frac{\kappa A}{k^2 - m_{\mathrm{A}}^2}\left(\bar{k}^2\delta^\mu_\alpha - \bar{k}^\mu\bar{k}_\alpha\right) + \frac{A}{\alpha}k^\mu k_\alpha - \\
&\quad - Ak^\mu k_\alpha + Ak^\mu k_\alpha + \frac{\kappa A}{k^2 - m_{\mathrm{A}}^2}\left(\frac{\bar{k}^2}{k^2}\bar{k}^\mu k_\alpha - \frac{\bar{k}^2}{k^2}\bar{k}^\mu k_\alpha\right) - \frac{A}{\alpha}k^\mu k_\alpha + \\
&\quad + Bk^2\bar{\delta}^\mu_\alpha - Bk^\mu\bar{k}_\alpha - \frac{\kappa B}{k^2 - m_{\mathrm{A}}^2}\left(\bar{k}^2\delta^\mu_\alpha - \bar{k}^\mu\bar{k}_\alpha\right) + \frac{B}{\alpha}k^\mu\bar{k}_\alpha + \\
&\quad + Ck^2\bar{k}^\mu\bar{k}_\alpha - C\bar{k}^2k^\mu\bar{k}_\alpha - \frac{\kappa C}{k^2 - m_{\mathrm{A}}^2}\left(\bar{k}^2\bar{k}^\mu\bar{k}_\alpha - \bar{k}^2\bar{k}^\mu\bar{k}_\alpha\right) + \frac{C}{\alpha}\bar{k}^2k^\mu\bar{k}_\alpha + \\
&\quad + D\frac{k^\mu k_\alpha}{k^2} - D\frac{k^\mu k_\alpha}{k^2} - \frac{\kappa D}{k^2 - m_{\mathrm{A}}^2}\left(\frac{\bar{k}^2}{(k^2)^2}\bar{k}^\mu k_\alpha - \frac{\bar{k}^2}{(k^2)^2}\bar{k}^\mu k_\alpha\right) + \frac{D}{\alpha}\frac{k^\mu k_\alpha}{k^2}.
\end{aligned}$$

Now comparing terms with the same tensorial character

$$\begin{aligned}
\delta^\mu_\alpha &: Ak^2 = 1 \\
k^\mu k_\alpha &: \frac{D}{\alpha}\frac{1}{k^2} = A \\
\bar{\delta}^\mu_\alpha &: Bk^2 - \frac{\kappa(A+B)\bar{k}^2}{k^2 - m_{\mathrm{A}}^2} = 0 \\
k^\mu\bar{k}_\alpha &: -B - C\bar{k}^2 = 0 \\
\bar{k}^\mu\bar{k}_\alpha &: Ck^2 - \frac{\kappa(B+A)}{k^2 - m_{\mathrm{A}}^2} = 0\,.
\end{aligned}$$

An independent subset of (four of) these gives (consistently with the remaining equation) the final result for the coefficients:

$$\begin{aligned}
A &= \frac{1}{k^2} \\
B &= \frac{\kappa \bar{k}^2 A}{k^2\left(k^2 - m_{\mathrm{A}}^2\right) - \kappa \bar{k}^2} \\
&= \frac{\kappa \bar{k}^2}{k^2\left(k^2\left(k^2 - m_{\mathrm{A}}^2\right) - \kappa \bar{k}^2\right)} \\
C &= -\frac{\kappa}{k^2\left(k^2\left(k^2 - m_{\mathrm{A}}^2\right) - \kappa \bar{k}^2\right)} \\
D &= \alpha\,,
\end{aligned}$$

so that we can finally write

$$\begin{aligned}
\mathcal{D}^{\mu\nu}(k;\alpha) = \frac{1}{k^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \frac{\kappa \bar{k}^2 \bar{g}^{\mu\nu}}{k^2\left(k^2\left(k^2 - m_{\mathrm{A}}^2\right) - \kappa \bar{k}^2\right)} - \\
- \frac{\kappa \bar{k}^\mu \bar{k}^\nu}{k^2\left(k^2\left(k^2 - m_{\mathrm{A}}^2\right) - \kappa \bar{k}^2\right)} + \alpha \frac{k^\mu k^\nu}{(k^2)^2}.
\end{aligned}$$

C. Explicit computation of the vacuum energy integral

We concentrate in this appendix on the computation of the contribution $I(k, \bar{k}; \kappa, m)$ of equation (4.3). Of course since the integral is divergent, it must be properly regularized and we choose to do that by putting an infrared (ϵ) and an ultraviolet (Λ) cutoff on the modulus of the momentum k and of its projection $\bar{k}$, exploiting some of the arbitrariness in the choice of the regularization scheme. Thus integrals written with implicit integration domain, like $\int d^4k(\ldots)$, are to be understood as performed in the domain of the variables (k_0, k_1, k_2, k_3), which is the inverse image of the domain $\epsilon \leq r \leq \Lambda$, $0 \leq \vartheta < 2\pi$, $\epsilon \leq \rho \leq \Lambda$, $0 \leq \varpi < 2\pi$ in the variables $(r, \rho, \vartheta, \varpi)$ under the following change of variables in *euclidean* space:

$$\begin{pmatrix} k^4 \\ k^1 \\ k^2 \\ k^3 \end{pmatrix} = T(r, \vartheta, \rho, \varpi) = \begin{pmatrix} r\cos\vartheta \\ r\sin\vartheta \\ \rho\cos\varpi \\ \rho\sin\varpi \end{pmatrix}$$

with jacobean

$$\mathcal{J}T = \begin{pmatrix} \cos\vartheta & -r\sin\vartheta & 0 & 0 \\ \sin\vartheta & r\cos\vartheta & 0 & 0 \\ 0 & 0 & \cos\varpi & -\rho\sin\varpi \\ 0 & 0 & \sin\varpi & \rho\cos\varpi \end{pmatrix},$$

whose determinant is

$$\det\left(\mathcal{J}T\right) = r\rho$$

so that

$$d^4k = r\rho\, dr\, d\rho\, d\vartheta\, d\varpi\,.$$

We thus have

$$
\begin{aligned}
&I(k,\bar{k};\kappa,m_{\mathrm{A}}) = \\
&= \int \frac{d^4k}{(2\pi)^4} \ln\left[k^2(k^2-m_{\mathrm{A}}^2)-\kappa\bar{k}^2\right] \\
&= \frac{1}{(2\pi)^4}\int_0^{2\pi}\!d\vartheta\int_0^{2\pi}\!d\varpi\int_\epsilon^\Lambda\!dr\int_\epsilon^\Lambda\!d\rho\,\det(\mathcal{J}T)\times\ln\left[(r^2+\rho^2)(r^2+\rho^2-m_{\mathrm{A}}^2)-\kappa\rho^2\right] \\
&= \frac{1}{4\pi^2}\int_\epsilon^\Lambda\!dr\int_\epsilon^\Lambda\!d\rho\,r\rho\times\ln\left[r^4+(2\rho^2-m_{\mathrm{A}}^2)r^2+\rho^2(\rho^2-m_{\mathrm{A}}^2-\kappa)\right] \\
&\equiv \frac{1}{4\pi^2}\,J(r,\rho;\kappa,m_{\mathrm{A}})\big|_{r,\rho=\epsilon}^{r,\rho=\Lambda}\,.
\end{aligned}
\tag{C.1}
$$

We have thus the problem of computing $J(r,\rho;\kappa,m_{\mathrm{A}})$, which can be calculated in closed form: the final result can be expressed as

$$
\begin{aligned}
J(r,\rho;\kappa,m_{\mathrm{A}}) = {}& \frac{\rho^2\left[\kappa+3(m_{\mathrm{A}}^2-6r^2-\rho^2)\right]}{24} + \\
&+\frac{\boldsymbol{B}^{3/2}}{48\kappa}\ln\left(\frac{\boldsymbol{C}-\sqrt{\boldsymbol{B}}}{\boldsymbol{C}+\sqrt{\boldsymbol{B}}}\right)-\frac{\boldsymbol{A}^{3/2}}{48\kappa}\ln\left(\frac{\kappa+\boldsymbol{C}-\sqrt{\boldsymbol{A}}}{\kappa+\boldsymbol{C}+\sqrt{\boldsymbol{A}}}\right)+ \\
&+\frac{\left[\kappa^2+3\kappa(m_{\mathrm{A}}^2-2r^2)+3(m_{\mathrm{A}}^4+2\boldsymbol{D})\right]}{48}\ln\left(-\kappa\rho^2+\boldsymbol{D}\right)
\end{aligned}
\tag{C.2}
$$

if we set

$$
\begin{aligned}
\boldsymbol{A} &= A(r;m_{\mathrm{A}},\kappa)\equiv(\kappa+m_{\mathrm{A}}^2)^2-4\kappa r^2 \\
\boldsymbol{B} &= B(\rho;m_{\mathrm{A}},\kappa)\equiv 4\kappa\rho^2+m_{\mathrm{A}}^4 \\
\boldsymbol{C} &= C(r,\rho;m_{\mathrm{A}})\equiv m_{\mathrm{A}}^2-2(r^2+\rho^2) \\
\boldsymbol{D} &= D(r,\rho;m_{\mathrm{A}})\equiv(r^2+\rho^2)^2-m_{\mathrm{A}}^2(r^2+\rho^2)=\frac{\boldsymbol{C}^2-\boldsymbol{B}}{4}+\kappa\rho^2\,.
\end{aligned}
$$

The final stage consists in evaluating it in the infrared ($\epsilon\to 0$) and ultraviolet limits. In this case this is equivalent to the computation of

$$
\lim_{(r,\rho)\to(0,0)} J(r,\rho;\kappa,m_{\mathrm{A}})
$$

and the extraction of the divergent contribution in

$$
J(r=\Lambda,\rho=\Lambda;\kappa,m_{\mathrm{A}})
$$

when $\Lambda\to\infty$. This is done in the next subsections.

C.1 Infrared limit

In this subsection we consider the infrared limit and in the following we will use the symbol "$\cong$" to imply that two expression are equivalent in the infrared limit, i.e. they have the same limit. Moreover we will get rid of the square roots by means of the following results

$$\sqrt{1-\frac{4\kappa r^2}{(\kappa+m_{\rm A}^2)^2}} \cong 1-\frac{2\kappa r^2}{|\kappa+m_{\rm A}^2|^2} \tag{C.3}$$

$$\sqrt{1+\frac{4\kappa\rho^2}{m_{\rm A}^4}} \cong 1+\frac{2\kappa\rho^2}{m_{\rm A}^4}. \tag{C.4}$$

We now turn to the contributions in the various lines of equation (C.2). The first one gives no problem:

$$\frac{\rho^2\left[\kappa+3(m_{\rm A}^2-6r^2-\rho^2)\right]}{24} \cong 0\,. \tag{C.5}$$

The second one has a well behavior in the $\boldsymbol{B}$ term; more care has to be paid in the logarithm:

$$\begin{aligned}
\frac{\boldsymbol{B}^{3/2}}{48\kappa}\ln\left(\frac{\boldsymbol{C}-\sqrt{\boldsymbol{B}}}{\boldsymbol{C}+\sqrt{\boldsymbol{B}}}\right) &\cong \frac{m_{\rm A}^6}{48\kappa}\ln\left[\frac{m_{\rm A}^2-2(r^2+\rho^2)-\sqrt{4\kappa\rho^2+m_{\rm A}^4}}{m_{\rm A}^2-2(r^2+\rho^2)+\sqrt{4\kappa\rho^2+m_{\rm A}^4}}\right]\\
&\cong \frac{m_{\rm A}^6}{48\kappa}\ln\left[\frac{m_{\rm A}^2-2(r^2+\rho^2)-m_{\rm A}^2\left(1+\frac{2\kappa\rho^2}{m_{\rm A}^4}\right)}{m_{\rm A}^2-2(r^2+\rho^2)+m_{\rm A}^2\left(1+\frac{2\kappa\rho^2}{m_{\rm A}^4}\right)}\right]\\
&\cong \frac{m_{\rm A}^6}{48\kappa}\ln\left[-m_{\rm A}^2(r^2+\rho^2)-\kappa\rho^2\right]-\frac{m_{\rm A}^6}{48\kappa}\ln\left[(m_{\rm A}^2)^2\right].
\end{aligned} \tag{C.6}$$

The last term on the second line of (C.2) again gives troubles only inside the logarithmic term, which can be elaborated as follows

$$\begin{aligned}
\ln\left(\frac{\kappa+\boldsymbol{C}-\sqrt{\boldsymbol{A}}}{\kappa+\boldsymbol{C}+\sqrt{\boldsymbol{A}}}\right) &= \ln\left[\frac{\kappa+m_{\rm A}^2-2(r^2+\rho^2)-\sqrt{(\kappa+m_{\rm A}^2)^2-4\kappa r^2}}{\kappa+m_{\rm A}^2-2(r^2+\rho^2)+\sqrt{(\kappa+m_{\rm A}^2)^2-4\kappa r^2}}\right]\\
&\cong \ln\left[\frac{\kappa+m_{\rm A}^2-2(r^2+\rho^2)-|\kappa+m_{\rm A}^2|\left(1-\frac{2\kappa r^2}{|\kappa+m_{\rm A}^2|^2}\right)}{\kappa+m_{\rm A}^2-2(r^2+\rho^2)+|\kappa+m_{\rm A}^2|\left(1-\frac{2\kappa r^2}{|\kappa+m_{\rm A}^2|^2}\right)}\right]\\
&\cong \begin{cases} \ln\left[\frac{-(r^2+\rho^2)+\frac{\kappa r^2}{\kappa+m_{\rm A}^2}}{(\kappa+m_{\rm A}^2)-(r^2+\rho^2)-\frac{\kappa r^2}{\kappa+m_{\rm A}^2}}\right] & \text{if } \kappa+m_{\rm A}^2>0\\ \ln\left[\left(\frac{-(r^2+\rho^2)+\frac{\kappa r^2}{\kappa+m_{\rm A}^2}}{(\kappa+m_{\rm A}^2)-(r^2+\rho^2)-\frac{\kappa r^2}{\kappa+m_{\rm A}^2}}\right)^{-1}\right] & \text{if } \kappa+m_{\rm A}^2<0 \end{cases}\\
&\cong \ln\left[\left(\frac{-(r^2+\rho^2)+\frac{\kappa r^2}{\kappa+m_{\rm A}^2}}{(\kappa+m_{\rm A}^2)-(r^2+\rho^2)-\frac{\kappa r^2}{\kappa+m_{\rm A}^2}}\right)^{\mathrm{Sign}(\kappa+m_{\rm A}^2)}\right]\\
&\cong \mathrm{Sign}(\kappa+m_{\rm A}^2)\left\{\ln\left[-m_{\rm A}^2(r^2+\rho^2)-\kappa\rho^2\right]-\ln\left[(\kappa+m_{\rm A}^2)^2\right]\right\}.
\end{aligned} \tag{C.7}$$

Since we also have

$$\frac{\boldsymbol{A}^{3/2}}{48\kappa} \cong \frac{|\kappa+m_{\rm A}^2|^3}{48\kappa}\,, \tag{C.8}$$

the two previous results, (C.8) and (C.7), combine in a neat way: the sign in the first factor exactly combines with the absolute value of the second factor

$$|\kappa + m_{\rm A}^2|^3 \mathrm{Sign}(\kappa + m_{\rm A}^2)\,,$$

so that

$$\begin{aligned} \frac{\boldsymbol{A}^{3/2}}{48\kappa} \ln\left(\frac{\kappa + \boldsymbol{C} - \sqrt{\boldsymbol{A}}}{\kappa + \boldsymbol{C} + \sqrt{\boldsymbol{A}}}\right) \cong & \frac{(\kappa + m_{\rm A}^2)^3}{48\kappa} \ln\left[-m_{\rm A}^2(r^2+\rho^2) - \kappa\rho^2\right] - \\ & - \frac{(\kappa + m_{\rm A}^2)^3}{48\kappa} \ln\left[(\kappa + m_{\rm A}^2)^2\right]. \end{aligned} \tag{C.9}$$

For the last term in (C.2) we do not have too much work. The factor before the logarithm has no problems and we can simply forget about the r and ρ dependent parts. Instead inside the logarithm we can neglect higher order terms in the limit we are interested in, so that

$$\begin{aligned} & \frac{\left[\kappa^2 + 3\kappa(m_{\rm A}^2 - 2r^2) + 3(m_{\rm A}^4 + 2\boldsymbol{D})\right]}{48} \ln\left(-\kappa\rho^2 + \boldsymbol{D}\right) \cong \\ & \cong \frac{\kappa^2 + 3\kappa m_{\rm A}^2 + 3m_{\rm A}^4}{48} \ln\left[-m_{\rm A}^2(r^2+\rho^2) - \kappa\rho^2\right]. \end{aligned} \tag{C.10}$$

The desired result, $J(0,0;\kappa,m_{\rm A})$ is then (C.5) + (C.6) − (C.9) + (C.10), i.e.

$$\begin{aligned} J(0,0;\kappa,m_{\rm A}) = & \frac{m_{\rm A}^6}{48\kappa}\left\{\ln\left[-m_{\rm A}^2(r^2+\rho^2) - \kappa\rho^2\right] - \ln\left[(m_{\rm A}^2)^2\right]\right\} + \\ & - \frac{(\kappa + m_{\rm A}^2)^3}{48\kappa} \ln\left[-m_{\rm A}^2(r^2+\rho^2) - \kappa\rho^2\right] + \\ & + \frac{(\kappa + m_{\rm A}^2)^3}{48\kappa} \ln\left[(\kappa + m_{\rm A}^2)^2\right] + \\ & + \frac{\kappa^2 + 3\kappa m_{\rm A}^2 + 3m_{\rm A}^4}{48} \ln\left[-m_{\rm A}^2(r^2+\rho^2) - \kappa\rho^2\right] \\ = & \frac{(\kappa + m_{\rm A}^2)^3}{24\kappa} \ln\left(\kappa + m_{\rm A}^2\right) - \frac{(m_{\rm A}^2)^3}{24\kappa} \ln\left(m_{\rm A}^2\right). \end{aligned} \tag{C.11}$$

As also pointed out in the main text this contribution is *finite* in the case of vanishing external field ($\kappa \to 0$), since

$$\lim_{\kappa\to 0}\left[\frac{(\kappa + m_{\rm A}^2)^3}{24\kappa} \ln\left(\kappa + m_{\rm A}^2\right) - \frac{(m_{\rm A}^2)^3}{24\kappa} \ln\left(m_{\rm A}^2\right)\right] = \frac{m_{\rm A}^4}{24}\left[\ln(m_{\rm A}^6) + 1\right]. \tag{C.12}$$

C.2 Ultraviolet limit

To tackle the problem of the ultraviolet behavior of the energy density, we analyze the limit in which $r \to \infty$, $\rho \to \infty$. As already discussed we will first set $r = \rho = \Lambda$ and then approximate the various quantities as $\Lambda \to \infty$. For the relevant expressions, already encountered above, we get[3]

$$\boldsymbol{A} = -4\kappa\Lambda^2\left(1 - \frac{(\kappa + m_{\rm A}^2)^2}{4\kappa\Lambda^2}\right), \quad \boldsymbol{A}^2 \sim 16\kappa^2\Lambda^4\left(1 - \frac{(\kappa + m_{\rm A}^2)^2}{2\kappa\Lambda^2}\right), \quad \ldots \tag{C.13}$$

[3]We now use the $\sim$ symbol to imply *the same behavior in the ultraviolet limit.*

$$
\boldsymbol{B} = 4\kappa\Lambda^2\left(1+\frac{m_{\rm A}^4}{4\kappa\Lambda^2}\right), \quad \boldsymbol{B}^2 \sim 16\kappa^2\Lambda^4\left(1+\frac{m_{\rm A}^4}{2\kappa\Lambda^2}\right), \quad \dots \tag{C.14}
$$

$$
\begin{aligned}
\boldsymbol{C} &= -4\Lambda^2\left(1-\frac{m_{\rm A}^2}{4\Lambda^2}\right), \quad \kappa+\boldsymbol{C} = -4\Lambda^2\left(1-\frac{\kappa+m_{\rm A}^2}{4\Lambda^2}\right), \\
&\qquad (\kappa+\boldsymbol{C})^{-1} \sim -\frac{1}{4\Lambda^2}\left(1+\frac{\kappa+m_{\rm A}^2}{4\Lambda^2}\right), \quad \dots
\end{aligned} \tag{C.15}
$$

$$
\begin{aligned}
\boldsymbol{D} &= 4\Lambda^4\left(1-\frac{m_{\rm A}^2}{2\Lambda^2}\right), \\
&\qquad \log(-\kappa\,\rho^2+\boldsymbol{D}) \sim \log(4\Lambda^4) - \frac{\kappa+2m_{\rm A}^2}{4\Lambda^2} - \frac{(\kappa+2m_{\rm A}^2)^4}{32\Lambda^2}\,.
\end{aligned} \tag{C.16}
$$

Moreover we also have the well known expansions

$$
\mathrm{Arctanh}\,(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \mathcal{O}(x^7) \tag{C.17}
$$

$$
\arctan\,(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \mathcal{O}(x^7)\,, \tag{C.18}
$$

which we are going to use in the following. In particular we can consider in generality the expansion of the following expression

$$
\begin{aligned}
-\frac{w^{3/2}}{2}\ln\left(\frac{z-w^{1/2}}{z+w^{1/2}}\right) &= w^{3/2}\mathrm{Arctanh}\left(\frac{w^{1/2}}{z}\right) \\
&= \begin{cases} |w|^{3/2}\mathrm{Arctanh}\left(\dfrac{|w|^{1/2}}{z}\right) & \text{if } w>0 \\ \qquad\qquad \Rightarrow w^{1/2} = |w|^{1/2} & \text{and} \quad w^{3/2} = |w|^{3/2} \\ -\imath|w|^{3/2}\mathrm{Arctanh}\left(\dfrac{\imath|w|^{1/2}}{z}\right) & \text{if } w<0 \\ \qquad\qquad \Rightarrow w^{1/2} = \imath|w|^{1/2} & \text{and} \quad w^{3/2} = -\imath|w|^{3/2} \end{cases} \\
&= \begin{cases} |w|^{3/2}\mathrm{Arctanh}\left(\dfrac{|w|^{1/2}}{z}\right) & \text{if } w>0 \\ -\imath|w|^{3/2}\imath\arctan\left(\dfrac{|w|^{1/2}}{z}\right) & \text{if } w<0 \end{cases} \\
&= \begin{cases} |w|^{3/2}\mathrm{Arctanh}\left(\dfrac{|w|^{1/2}}{z}\right) & \text{if } w>0 \\ |w|^{3/2}\arctan\left(\dfrac{|w|^{1/2}}{z}\right) & \text{if } w<0 \end{cases} \\
&= |w|^{3/2}\left[\left(\frac{|w|^{1/2}}{z}\right) - \mathrm{sign}(w)\frac{1}{3}\left(\frac{|w|^{1/2}}{z}\right)^3 + \right. \\
&\qquad \left. + \frac{1}{5}\left(\frac{|w|^{1/2}}{z}\right)^5 - \mathrm{sign}(w)\left(\frac{|w|^{1/2}}{z}\right)^7 + \cdots\right],
\end{aligned}
$$

so that

$$-\frac{w^{3/2}}{2}\ln\left(\frac{z-w^{1/2}}{z+w^{1/2}}\right)=w^2\left[\sum_n^{0,\infty}\frac{w^n}{(2n+1)z^{2n+1}}\right]. \tag{C.19}$$

From the above result, if we identify

$$w\longleftrightarrow \boldsymbol{B}\,,$$
$$z\longleftrightarrow \boldsymbol{C}$$

and we consider an overall $1/(24\kappa)$ factor, using properly (C.14) and the first equation of (C.15), we get

$$\frac{\boldsymbol{B}^{3/2}}{48k}\ln\left(\frac{\boldsymbol{C}-\sqrt{\boldsymbol{B}}}{\boldsymbol{C}+\sqrt{\boldsymbol{B}}}\right)\sim\frac{\kappa\Lambda^2}{6}+\frac{\kappa^2+3\kappa m^2+6m^4}{72} \tag{C.20}$$

for the second term in (C.2). In the same way, starting again from result (C.19), together with the identifications

$$w\longleftrightarrow \boldsymbol{A}\,,$$
$$z\longleftrightarrow \kappa+\boldsymbol{C}$$

and taking into account an overall $-1/(24\kappa)$, (C.13) and (C.15), we obtain for the third term in (C.2)

$$-\frac{A^{3/2}}{48\kappa}\ln\left(\frac{\kappa+\boldsymbol{C}-\sqrt{\boldsymbol{A}}}{\kappa+\boldsymbol{C}+\sqrt{\boldsymbol{A}}}\right)\sim-\frac{\kappa\Lambda^2}{6}+\frac{4\kappa^2+9\kappa m^2+6m^4}{72}\,. \tag{C.21}$$

The last term in (C.2) has also a logarithmic part and, using (C.16), can be approximated as

$$\begin{aligned}\frac{1}{48}&\left[\kappa^2+3\kappa m_{\mathrm{A}}^2-6\kappa r^2+3m_{\mathrm{A}}^4+6\boldsymbol{D}\right]\ln\left(-\kappa\rho^2+\boldsymbol{D}\right)\sim\\ &\sim\frac{\kappa^2+3\kappa m_{\mathrm{A}}^2+3m_{\mathrm{A}}^4}{48}\ln\left(4\Lambda^4\right)-\frac{\kappa+2m_{\mathrm{A}}^2}{8}\Lambda^2\ln\left(4\Lambda^4\right)+\frac{\Lambda^4}{2}\ln\left(4\Lambda^4\right)+\\ &\quad+\frac{(\kappa+2m_{\mathrm{A}}^2)^2}{64}-\frac{\kappa+2m_{\mathrm{A}}^2}{8}\Lambda^2\,. \end{aligned}\tag{C.22}$$

We are now concerned with the easiest term in (C.2), namely the first, for which we get

$$\frac{\rho^2\left(\kappa+3\left(m_{\mathrm{A}}^2-6r^2-\rho^2\right)\right)}{24}\sim\frac{\kappa+3m_{\mathrm{A}}^2}{24}\Lambda^2-\frac{7}{8}\Lambda^4\,. \tag{C.23}$$

Summing up equations (C.20), (C.21), (C.22), (C.23), we obtain

$$\begin{aligned}&\Lambda^4\left[2\ln\Lambda+\ln 2-\frac{7}{8}\right]+\Lambda^2\left[-\frac{2\kappa+3m_{\mathrm{A}}^2}{24}-\frac{\kappa+2m_{\mathrm{A}}^2}{4}\left(2\ln\Lambda+\ln 2\right)\right]+\\ &+\frac{5\kappa^2+12\kappa m_{\mathrm{A}}^2+12m_{\mathrm{A}}^4}{72}+\frac{(\kappa+2m_{\mathrm{A}}^2)^2}{64}+\frac{\kappa^2+3\kappa m_{\mathrm{A}}^2+3m_{\mathrm{A}}^4}{48}\left(2\ln\Lambda+\ln 2\right).\end{aligned}\tag{C.24}$$

C.3 Result for $I(r=\Lambda,\rho=\Lambda;\kappa,m_{\rm A}) - \lim_{(r,\rho)\to(0,0)} I(r,\rho;\kappa,m_{\rm A})$

From the results (C.11), (C.24) of the two previous subsections we obtain what we are interested in,

$$
\begin{aligned}
& J(r=\Lambda,\rho=\Lambda;\kappa,m_{\rm A}) - \lim_{(r,\rho)\to(0,0)} J(r,\rho;\kappa,m_{\rm A}) \sim \\
& \sim \Lambda^4\left[2\ln\Lambda + \ln 2 - \frac{7}{8}\right] + \Lambda^2\left[-\frac{2\kappa+3m_{\rm A}^2}{24} - \frac{\kappa+2m_{\rm A}^2}{4}(2\ln\Lambda+\ln 2)\right] + \\
& \quad + \frac{5\kappa^2+12\kappa m_{\rm A}^2+12m_{\rm A}^4}{72} + \frac{(\kappa+2m_{\rm A}^2)^2}{64} + \frac{\kappa^2+3\kappa m_{\rm A}^2+3m_{\rm A}^4}{48}(2\ln\Lambda+\ln 2) - \\
& \quad - \frac{(\kappa+m_{\rm A}^2)^3}{24\kappa}\ln\left(\kappa+m_{\rm A}^2\right) + \frac{(m_{\rm A}^2)^3}{24\kappa}\ln\left(m_{\rm A}^2\right). \\
& \sim \left(I_\Lambda^{(4)} - \ln 2 + \frac{3}{4}\right)\Lambda^4 + (I_{\ln\Lambda}^{(4)} - 2)\Lambda^4\ln\Lambda + I_\Lambda^{(2)}\Lambda^2 + I_{\ln\Lambda}^{(2)}\Lambda^2 + I^{(0)} + I_{\ln\Lambda}^{(0)}, \qquad \text{(C.25)}
\end{aligned}
$$

where for convenience in the last line we have used the following definitions:

$$
\begin{aligned}
I_\Lambda^{(4)} &= 2\ln 2 - \frac{13}{8} \\
I_{\ln\Lambda}^{(4)} &= 4 \\
I_\Lambda^{(2)} &= -\frac{2\kappa+3m_{\rm A}^2}{24} - \frac{\kappa+2m_{\rm A}^2}{4}\ln 2 \\
I_{\ln\Lambda}^{(2)} &= -\frac{\kappa+2m_{\rm A}^2}{2} \\
I^{(0)} &= \frac{49\kappa^2+132\kappa m_{\rm A}^2+132m_{\rm A}^4}{576} + \frac{\kappa^2+3\kappa m_{\rm A}^2+3m_{\rm A}^4}{48}\ln 2 - \\
&\quad - \frac{(\kappa+m_{\rm A}^2)^3}{24\kappa}\ln\left(\kappa+m_{\rm A}^2\right) + \frac{(m_{\rm A}^2)^3}{24\kappa}\ln\left(m_{\rm A}^2\right) \\
I_{\ln\Lambda}^{(0)} &= \frac{\kappa^2+3\kappa m_{\rm A}^2+3m_{\rm A}^4}{24}. \qquad \text{(C.26)}
\end{aligned}
$$

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